



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE THEORY OF REVERSIONS.*

SQUARES like those shown in Figs. 1 and 2, in which the numbers occur in their natural order, are known as *natural squares*. In such squares, it will be noticed that the numbers in associated¹ cells are complementary, i. e., their sum is twice the mean number. It follows that any two columns equally distant from the central bar of the lattice are complementary columns, that is, the magic sum

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Fig. 1.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

Fig. 2.

will be the mean of their sums. Further any two numbers in these complementary columns which lie in the same row have a constant difference, and therefore the sums of the two columns differ by n times this difference. If then we raise the lighter column and depress the heavier column by $n/2$ times this difference we shall bring both to the

* This paper was extracted about 18 months ago from three different parts of an unpublished treatise written in 1894. With regard to footnote 6, p. 63, since this was written Sayles and Worthington have independently solved the problem of construction for 6'.

¹ Two cells are said to be *associated* when the straight line joining their centers intersects the center of the lattice, and they are equally distant from that center.

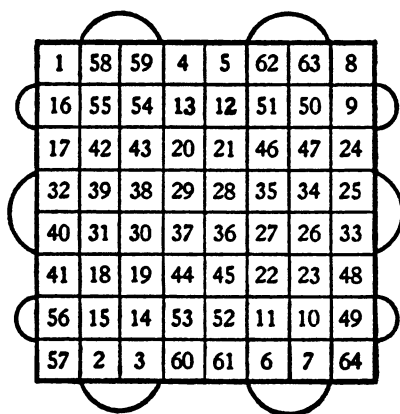
mean value. Now we can effect this change by interchanging half the numbers in the one column with the numbers in the other column lying in their respective rows. The same is true with regard to rows, so that if we can make $n/2$ horizontal interchanges between every pair of complementary columns and the same number of vertical interchanges between every pair of complementary rows, we shall have the magic sum in all rows and columns. It is easy to see that we can do this by reversing half the rows and half the columns, provided the two operations are so arranged as not to interfere with one another. This last condition can be assured by always turning over columns and rows in associated pairs, for then we shall have made horizontal interchanges only between pairs of numbers previously untouched or between pairs, each of whose constituents has already received an equal vertical displacement; and similarly with the vertical interchanges. By this method, it will be noticed, we always secure magic central diagonals, for however we choose our rows and columns we only alter the central diagonals of the natural square (which are already magic) by interchanging pairs of complementaries with other pairs of complementaries.

Since the $n/2$ columns have to be arranged in pairs on either side of the central vertical bar of the lattice, $n/2$ must be even, and so the method, *in its simplest form*, applies only to orders $\equiv 0 \pmod{4}$. We may formulate the rule thus: *For orders of form $4m$, reverse m pairs of complementary columns and m pairs of complementary rows, and the crude magic is completed.*

In the following example the curved lines indicate the rows and columns which have been reversed (Fig. 3).

We have said that this method applies only when $n/2$ is even, but we shall now show that by a slight modification it can be applied to all even orders. For suppose n is double-of-odd; we cannot then arrange half the columns

in pairs about the center since their number is odd, but we can so arrange $n/2-1$ rows and $n/2-1$ columns, and if we reverse all these rows and columns we shall have made $n/2-1$ interchanges between every pair of complementary rows and columns. We now require only to make the one further interchange between every pair of rows and columns, without interfering with the previous changes or with the central diagonals. To effect this is always



1	58	59	4	5	62	63	8
16	55	54	13	12	51	50	9
17	42	43	20	21	46	47	24
32	39	38	29	28	35	34	25
40	31	30	37	36	27	26	33
41	18	19	44	45	22	23	48
56	15	14	53	52	11	10	49
57	2	3	60	61	6	7	64

Fig. 3.

easy with any orders $\equiv 2 \pmod{4}$, (6, 10, 14 etc.), excepting the first. In the case of 6^2 an artifice is necessary. If we reverse the two central diagonals of a square it will be found, on examination, that this is equivalent to reversing two rows and two columns; in fact, this gives us a method of forming the magic 4^2 from the natural square with the least number of displacements, thus:

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Fig. 4

Applying this idea, we can complete the crude magic

6^2 from the scheme shown in Fig. 5, where horizontal lines indicate horizontal interchanges, and vertical lines vertical interchanges; the lines through the diagonals implying that the diagonals are to be reversed. The resulting magic is shown in Fig. 6.

The general method here described is known as the *method of reversions*, and the artifice used in the double-of-odd orders is called *the broken reversion*. The method of reversions, as applied to all even orders, both in squares and cubes, was first(?) investigated by the late W. Firth, Scholar of Emmanuel, Cambridge.²

The broken reversion for 6^2 may, of course, be made in various ways, but the above scheme is one of the most sym-

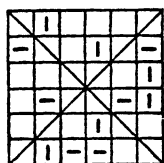


Fig. 5.

36	32	3	4	5	31
12	29	9	28	26	7
13	14	22	21	17	24
19	23	16	15	20	18
25	11	27	10	8	30
6	2	34	33	35	1

Fig. 6.

metrical, and may be memorialized thus: *For horizontal changes commence at the two middle cells of the bottom row, and progress upwards and divergently along two knight's paths. For vertical changes turn the square on one of its sides and proceed as before.*

In dealing with larger double-of-odd orders we may leave the central diagonals "intact" and invert $n/2-1$ rows and $n/2-1$ columns. The broken reversion can then always be effected in a multitude of ways. It must be kept in mind, however, that in making horizontal changes we must not touch numbers which have been already moved horizontally, and if we use a number which has received

² Died 1889. For historical notice *vide* section on cubes.

a vertical displacement we can only change it with a number which has received an equal vertical displacement, and similarly with vertical interchanges. Lastly we must not touch the central diagonals.

Fig. 7 is such a scheme for 10^2 , with the four central rows and columns reversed, and Fig. 8 shows the completed magic.

It is unnecessary to formulate a rule for making the reversions in these cases, because we are about to consider the method from a broader standpoint which will lead up to a general rule.

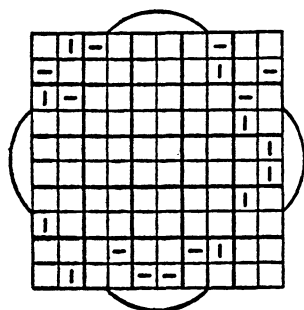


Fig. 7.

1	92	8	94	95	96	97	3	9	10
20	12	13	84	85	86	87	88	19	11
71	29	23	74	75	76	77	28	22	30
40	39	38	67	66	65	64	33	62	31
50	49	48	57	56	55	54	43	42	51
60	59	58	47	46	45	44	53	52	41
70	69	68	37	36	35	34	63	32	61
21	72	73	24	25	26	27	78	79	80
81	82	83	17	15	16	14	18	89	90
91	2	93	4	6	5	7	98	99	100

Fig. 8.

If the reader will consider the method used in forming the magic 6^2 by reversing the central diagonals, he will find that this artifice amounts to taking in every column two numbers equally distant from the central horizontal bar and interchanging each of them with its complementary in the associated cell, the operation being so arranged that two and only two numbers are moved in each row. This, as we have already pointed out, is equivalent to reversing two rows and two columns. Now these skew interchanges need not be made on the central diagonals—they can be made in any part of the lattice, provided the con-

ditions just laid down are attended to. If then we make a second series of skew changes of like kind, we shall have, in effect, reversed 4 rows and 4 columns, and so on, each complete skew reversion representing two rows and columns. Now if $n \equiv 2 \pmod{4}$ we have to reverse $n/2 - 1$ rows and columns before making the broken reversion, therefore the same result is attained by making $(n-2)/4$ complete sets of skew reversions and one broken reversion.

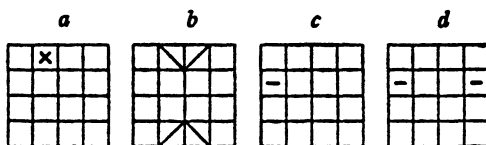


Fig. 9.

In like manner, if $n \equiv 0 \pmod{4}$, instead of reversing $n/2$ rows and columns we need only to make $n/4$ sets of skew reversions.

We shall define the symbol $[\times]$ as implying that skew interchanges are to be made between opposed pairs of the four numbers symmetrically situated with regard to the central horizontal and vertical bars, one of which numbers



Fig. 10.

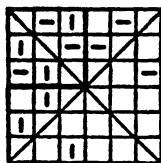


Fig. 11.

36	5	33	4	2	31
25	29	10	9	26	12
18	20	22	21	17	13
19	14	16	15	23	24
7	11	27	28	8	30
6	32	3	34	35	1

Fig. 12.

occupies the cell in which the symbol is placed. In other words we shall assume that Fig. 9a indicates what we have hitherto represented as in Fig. 9b. Further, it is quite unnecessary to use two symbols for a vertical or horizontal change, for Fig. 9c sufficiently indicates the same as Fig. 9d. If these abbreviations are granted, a scheme like Fig.

5 may be replaced by a small square like Fig. 10, which is to be applied to the top left-hand corner of the natural 6^2 .

Fig. 11 is the extended scheme from Fig. 10, and Fig. 12 is the resulting magic. The small squares of symbols like Fig. 10 may be called "*index squares*."

The law of formation for index squares is sufficiently obvious. To secure magic rows and columns in the resulting square, the symbols — and | must occur once on each row and column of the index, and the symbol \times an equal number of times on each row and column; that is, if there are two series $\times \times \dots \times$ the symbol \times must appear twice in every row and twice in every column, and so on. But we already know by the theory of paths that these conditions can be assured by laying the successive symbolic periods along parallel paths of the index, whose coordinates are prime to the order of the index. If we decide always to use parallel diagonal paths and always to apply the index to the top left-hand corner of the natural square, the index square will be completely represented by its top row. In Fig. 10 this is $\boxed{\times \mid - \mid}$, which we may call the index-rod of the square, or we may simply call Fig. 12 the magic $\boxed{\times \mid - \mid}$. Remembering that we require $(n-2)/4$ sets of skew reversions when $n \equiv 2 \pmod{4}$ and $n/4$ when $n \equiv 0$, it is obvious that the following rule will give crude magic squares of any even order n :

Take a rod of $n/2$ cells, $n/4$ symbols of the form \times , (using the integral part of $n/4$ only), and if there is a remainder when n is divided by 4, add the symbols | and —. Place one of the symbols \times in the left-hand cell of the rod, and the other symbols in any cell, but not more than one in each cell. The result is an index-rod for the magic n^2 .

Take a square lattice of order $n/2$, and lay the rod along the top row of the lattice. Fill up every diagonal slanting downward and to the right which has a symbol in its highest cell with repetitions of that symbol. The re-

sulting index-square if applied to the lop left-hand corner of the natural n^2 , with the symbols allowed the operative powers already defined, will produce the magic n^2 .

The following are index-rods for squares of even orders:

4^2	<table><tr><td>×</td><td></td></tr></table>	×		10^2	<table><tr><td>×</td><td></td><td> </td><td>×</td><td>-</td></tr></table>	×			×	-				
×														
×			×	-										
6^2	<table><tr><td>×</td><td>-</td><td> </td></tr></table>	×	-		12^2	<table><tr><td>×</td><td></td><td></td><td>×</td><td>×</td><td></td></tr></table>	×			×	×			
×	-													
×			×	×										
8^2	<table><tr><td>×</td><td></td><td>×</td><td></td></tr></table>	×		×		14^2	<table><tr><td>×</td><td>-</td><td>×</td><td></td><td></td><td>×</td><td> </td></tr></table>	×	-	×			×	
×		×												
×	-	×			×									

When the number of cells in the rod exceeds the number of symbols, as it always does excepting with 6^2 , the first cell may be left blank. Also, if there are sufficient blank cells, a \times may be replaced by two vertical and two horizontal symbols. Thus 12^2 might be given so

×			-	×	-
---	--	--	---	---	---

×			-	×	-
---	--	--	---	---	---

×			-	×	-
-	×			-	×
×	-	×			-
-	×	-	×		
	-	×	-	×	
		-	×	-	×

Fig. 13.

144	134	135	9	140	7	6	137	4	10	11	133
24	131	123	124	20	127	126	17	21	22	122	13
120	35	118	112	113	31	30	32	33	111	26	109
48	107	46	105	101	102	43	44	100	39	98	37
85	59	94	57	92	90	55	89	52	87	50	60
73	74	70	81	68	79	78	65	76	63	71	72
61	62	75	69	77	67	66	80	64	82	83	84
49	86	58	88	56	54	91	53	93	51	95	96
97	47	99	45	41	42	103	104	40	106	38	108
36	110	34	28	29	114	115	116	117	27	119	25
121	23	15	16	125	19	18	128	129	130	14	132
12	2	3	136	8	138	139	5	141	142	143	1

Fig. 14.

This presentation of 12^2 is shown in Figs. 13, 14, and 14^2 from the index-rod given above, in Figs. 15, 16.

Of course the employment of diagonal paths in the construction of the index is purely a matter of convenience. In the following index for 10^2 , (Fig. 17) the skew-symbols

are placed along two parallel paths (2, 1) and the symbols — and | are then added so that each shall appear once in each row and once in each column, but neither of them on the diagonal of the index slanting upward and to the left.

x	-	x			x	
---	---	---	--	--	---	--

x	-	x			x	
	x	-	x			x
x		x	-	x		
	x		x	-	x	
		x		x	-	x
x			x		x	-
-	x			x		x

Fig. 15.

196	13	194	4	5	191	189	8	188	10	11	185	2	183
169	181	26	179	19	20	176	175	23	24	172	17	170	28
168	156	166	39	164	34	35	36	37	159	32	157	41	155
43	153	143	151	52	149	49	50	146	47	144	54	142	56
57	58	138	130	136	65	134	133	62	131	67	129	69	70
126	72	73	123	117	121	78	77	118	80	116	82	83	113
98	111	87	88	108	104	106	105	93	103	95	96	100	85
99	97	101	102	94	90	92	91	107	89	109	110	86	112
84	114	115	81	75	79	119	120	76	122	74	124	125	71
127	128	68	60	66	132	64	63	135	61	137	59	139	140
141	55	45	53	145	51	147	148	48	150	46	152	44	154
42	30	40	158	38	160	161	162	163	33	165	31	167	29
15	27	171	25	173	174	22	21	177	178	18	180	16	182
14	184	12	186	187	9	7	190	6	192	193	3	195	1

Fig. 16.

Crude cubes of even orders we shall treat by the index-rod as in the section on squares. The reader will remember that we constructed squares of orders $\equiv 0 \pmod{4}$ by re-

x		x	-	
x		-	x	
	x		x	-
-	x			x
	-	x		x

Fig. 17.

versing half the rows and half the columns, and it is easy to obtain an analogous method for the cubes of the same family. Suppose we reverse half the V-planes^{*} in asso-

* P-plane = Presentation-, or Paper-plane; H-plane = Horizontal plane; V-plane = Vertical plane.

ciated pairs; that is, turn each through an angle of 180° round a horizontal axis parallel to the paper-plane so that the associated columns in each plane are interchanged and reversed. We evidently give to every row of the cube the magic sum, for half the numbers in each row will be ex-

1	62	63	4
5	58	59	8
9	54	55	12
13	50	51	16

17	46	47	20
21	42	43	24
25	38	39	28
29	34	35	32

33	30	31	36
37	26	27	40
41	22	23	44
45	18	19	48

49	14	15	52
53	10	11	56
57	6	7	60
61	2	3	64

Magic in rows only.

Fig. 18. The natural 4^3 with V-planes reversed.

1	62	63	4
56	11	10	53
60	7	6	57
13	50	51	16

17	46	47	20
40	27	26	37
44	23	22	41
29	34	35	32

33	30	31	36
24	43	42	21
28	39	38	25
45	18	19	48

49	14	15	52
8	59	58	5
12	55	54	9
61	2	3	64

Magic in rows and columns.

Fig. 19. Being Fig. 18 with H-planes reversed.

1	62	63	4
56	11	10	53
60	7	6	57
13	50	51	16

32	35	34	29
41	22	23	44
37	26	27	40
20	47	46	17

48	19	18	45
25	38	39	28
21	42	43	24
36	31	30	33

49	14	15	52
8	59	58	5
12	55	54	9
61	2	3	64

Magic in rows, columns and lines.

Fig. 20. Being Fig. 19, with P-planes reversed.

CRUDE MAGIC 4^3 .

changed for their complementaries. If we do likewise with H-planes and P-planes the rows and lines⁴ will become magic. But as with the square, and for like reasons, these three operations can be performed without mutual interference. Hence the simple general rule for all cubes of the double-of-even orders:

⁴"Line" = a contiguous series of cells measured at right angles to the paper-plane.

Reverse, in associated pairs, half the V-planes, half the H-planes, and half the P-planes.

With this method the central great diagonals, of course, maintain their magic properties, as they must do for the cube to be considered even a crude magic.⁵ To make the operation clear to the reader we append views of 4^3 at each

A	B	C																																																																																																																																																																																																						
<table><tr><td>2</td><td>8</td><td>6</td><td>1</td><td>2</td><td>8</td></tr><tr><td>⁵³₁₇</td><td></td><td>⁴⁷₈₃</td><td></td><td>⁵³₁₇</td><td></td></tr><tr><td>6</td><td>4</td><td>2</td><td>5</td><td>6</td><td>4</td></tr><tr><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>6</td><td>2</td><td>5</td><td>8</td><td>1</td><td>5</td></tr><tr><td>⁴⁸₇₃</td><td></td><td>²³₁₄</td><td></td><td>⁷³₄₈</td><td></td></tr><tr><td>1</td><td>5</td><td>6</td><td>7</td><td>6</td><td>2</td></tr><tr><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>8</td><td>2</td><td>6</td><td>1</td><td>8</td><td>2</td></tr><tr><td>³⁵₇₁</td><td></td><td>⁴⁷₈₃</td><td></td><td>³⁵₇₁</td><td></td></tr><tr><td>4</td><td>6</td><td>2</td><td>5</td><td>4</td><td>6</td></tr></table>	2	8	6	1	2	8	⁵³ ₁₇		⁴⁷ ₈₃		⁵³ ₁₇		6	4	2	5	6	4							6	2	5	8	1	5	⁴⁸ ₇₃		²³ ₁₄		⁷³ ₄₈		1	5	6	7	6	2							8	2	6	1	8	2	³⁵ ₇₁		⁴⁷ ₈₃		³⁵ ₇₁		4	6	2	5	4	6	<table><tr><td>5</td><td>2</td><td>2</td><td>8</td><td>6</td><td>4</td></tr><tr><td>⁸³₇₄</td><td></td><td>⁵³₁₇</td><td></td><td>⁷¹₅₃</td><td></td></tr><tr><td>6</td><td>1</td><td>6</td><td>4</td><td>8</td><td>2</td></tr><tr><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>6</td><td>2</td><td>5</td><td>2</td><td>8</td><td>4</td></tr><tr><td>³⁷₅₁</td><td></td><td>⁸³₇₄</td><td></td><td>¹⁵₇₃</td><td></td></tr><tr><td>4</td><td>8</td><td>6</td><td>1</td><td>2</td><td>6</td></tr><tr><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>2</td><td>8</td><td>6</td><td>4</td><td>1</td><td>6</td></tr><tr><td>³⁵₇₁</td><td></td><td>¹⁷₅₃</td><td></td><td>⁴⁷₃₈</td><td></td></tr><tr><td>4</td><td>6</td><td>2</td><td>8</td><td>2</td><td>5</td></tr></table>	5	2	2	8	6	4	⁸³ ₇₄		⁵³ ₁₇		⁷¹ ₅₃		6	1	6	4	8	2							6	2	5	2	8	4	³⁷ ₅₁		⁸³ ₇₄		¹⁵ ₇₃		4	8	6	1	2	6							2	8	6	4	1	6	³⁵ ₇₁		¹⁷ ₅₃		⁴⁷ ₃₈		4	6	2	8	2	5	<table><tr><td>6</td><td>4</td><td>2</td><td>5</td><td>6</td><td>4</td></tr><tr><td>¹⁷₅₃</td><td></td><td>⁸³₄₇</td><td></td><td>¹⁷₅₃</td><td></td></tr><tr><td>2</td><td>8</td><td>6</td><td>1</td><td>2</td><td>8</td></tr><tr><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>1</td><td>5</td><td>6</td><td>7</td><td>6</td><td>2</td></tr><tr><td>⁷³₄₈</td><td></td><td>¹⁴₂₃</td><td></td><td>⁴⁸₇₃</td><td></td></tr><tr><td>6</td><td>2</td><td>5</td><td>8</td><td>1</td><td>5</td></tr><tr><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>4</td><td>6</td><td>2</td><td>5</td><td>4</td><td>6</td></tr><tr><td>⁷¹₅₃</td><td></td><td>⁸³₄₇</td><td></td><td>⁷¹₅₃</td><td></td></tr><tr><td>8</td><td>2</td><td>6</td><td>1</td><td>8</td><td>2</td></tr></table>	6	4	2	5	6	4	¹⁷ ₅₃		⁸³ ₄₇		¹⁷ ₅₃		2	8	6	1	2	8							1	5	6	7	6	2	⁷³ ₄₈		¹⁴ ₂₃		⁴⁸ ₇₃		6	2	5	8	1	5							4	6	2	5	4	6	⁷¹ ₅₃		⁸³ ₄₇		⁷¹ ₅₃		8	2	6	1	8	2
2	8	6	1	2	8																																																																																																																																																																																																			
⁵³ ₁₇		⁴⁷ ₈₃		⁵³ ₁₇																																																																																																																																																																																																				
6	4	2	5	6	4																																																																																																																																																																																																			
6	2	5	8	1	5																																																																																																																																																																																																			
⁴⁸ ₇₃		²³ ₁₄		⁷³ ₄₈																																																																																																																																																																																																				
1	5	6	7	6	2																																																																																																																																																																																																			
8	2	6	1	8	2																																																																																																																																																																																																			
³⁵ ₇₁		⁴⁷ ₈₃		³⁵ ₇₁																																																																																																																																																																																																				
4	6	2	5	4	6																																																																																																																																																																																																			
5	2	2	8	6	4																																																																																																																																																																																																			
⁸³ ₇₄		⁵³ ₁₇		⁷¹ ₅₃																																																																																																																																																																																																				
6	1	6	4	8	2																																																																																																																																																																																																			
6	2	5	2	8	4																																																																																																																																																																																																			
³⁷ ₅₁		⁸³ ₇₄		¹⁵ ₇₃																																																																																																																																																																																																				
4	8	6	1	2	6																																																																																																																																																																																																			
2	8	6	4	1	6																																																																																																																																																																																																			
³⁵ ₇₁		¹⁷ ₅₃		⁴⁷ ₃₈																																																																																																																																																																																																				
4	6	2	8	2	5																																																																																																																																																																																																			
6	4	2	5	6	4																																																																																																																																																																																																			
¹⁷ ₅₃		⁸³ ₄₇		¹⁷ ₅₃																																																																																																																																																																																																				
2	8	6	1	2	8																																																																																																																																																																																																			
1	5	6	7	6	2																																																																																																																																																																																																			
⁷³ ₄₈		¹⁴ ₂₃		⁴⁸ ₇₃																																																																																																																																																																																																				
6	2	5	8	1	5																																																																																																																																																																																																			
4	6	2	5	4	6																																																																																																																																																																																																			
⁷¹ ₅₃		⁸³ ₄₇		⁷¹ ₅₃																																																																																																																																																																																																				
8	2	6	1	8	2																																																																																																																																																																																																			

Fig. 21.

separate stage, the central pair of planes being used at each reversion.

By this method the reader can make any crude magic cube of order $4m$. With orders of form $4m+2$ we find the same difficulties as with squares of like orders. So far as we are aware no magic cube of this family had been

1	17	24
23	3	16
18	22	2

15	19	8
7	14	21
20	9	13

26	6	10
12	25	5
4	11	27

Fig. 22.

constructed until Firth succeeded with 6^3 in 1889, and we believe those we shall presently construct are the first which have been published.⁶ Firth's original cube was built up by the method of "pseudo-cubes," being an extension to solid magics of Thompson's method. The cube of 216 cells was divided into 27 subsidiary cubes each con-

⁵ A cube which is faulty on one of its central great diagonals is no more a magic than is a square which is faulty on one of its central diagonals.

⁶ The recent examples published by Willis and Kingery fail in their central great diagonals, a fatal defect.

taining 2 cells in an edge. The 8 cells of each subsidiary were filled with the numbers 1 to 8 in such a way that each row, column, line, and *central great diagonal* of the large cube summed 27. The cube was then completed by using the magic 3^3 in the same way that 6^2 is constructed from 3^2 . Firth formulated no rule for arrangement of the numbers in the pseudo-cubes, and great difficulty was encountered in balancing the central great diagonals. His pseudo-

I						II						III					
2	8	134	129	186	192	5	3	132	135	189	187	117	114	146	152	62	60
6	4	130	133	190	188	1	7	136	131	185	191	118	113	150	148	64	58
182	178	21	24	121	125	180	184	18	19	127	123	54	50	109	106	168	164
177	181	22	23	126	122	183	179	17	20	124	128	52	56	110	105	162	166
144	138	174	169	16	10	139	141	172	175	11	13	154	160	70	68	97	102
140	142	170	173	12	14	143	137	176	171	15	9	156	158	66	72	98	101

IV						V						VI					
120	115	149	147	63	59	206	204	42	45	78	76	201	207	48	43	73	79
119	116	145	151	61	59	202	208	46	41	74	80	205	203	44	47	77	75
51	55	112	107	161	165	89	93	198	199	38	34	95	91	193	196	36	40
53	49	111	108	167	163	94	90	197	200	33	37	92	96	194	195	39	35
155	157	65	71	100	103	28	30	82	85	212	214	31	25	88	83	215	209
153	159	69	67	99	104	32	26	86	81	216	210	27	29	84	87	211	213

Fig. 23.

skeleton is shown in Fig. 21, where each plate represents two P-planes of 6^3 , each plate containing 9 pseudo-cubes. The numbers in the subsidiaries are shown in diagrammatic perspective, the four "larger" numbers lying in the anterior layer, and the four "smaller" numbers, grouped in the center, in the posterior layer.

If we use this with the magic of Fig. 22 we obtain the magic 6^3 shown in Fig. 23.

This cube is non-La Hireian, as is frequently the case with magics constructed by this method.

The scheme of pseudo-cubes for 6^3 once found, we can easily extend the method to any double-of-odd order in the following manner. Take the pseudo-scheme of next lower order [e. g., 6^3 to make 10^3 , 10^3 to make 14^3 etc.]. To each of three outside plates of cubes, which meet at any corner of the skeleton, apply a replica-plate, and to each of the other three faces a complementary to the plate opposed to it, that is a plate in which each number replaces its complementary number (1 for 8, 2 for 7, etc.). We now have a properly balanced skeleton for the next double-of-odd order, wanting only its 12 edges. Consider any three edges that meet at a corner of the cube; they can be completed (wanting their corner-cubes) by placing in each of them any row of cubes from the original skeleton. Each of these three edges has three other edges parallel to it, two lying in the same square planes with it and the third diagonally opposed to it. In the former we may place edges complementary to the edge to which they are parallel, and in the latter a replica of the same. The skeleton wants now only its 8 corner pseudo-cubes. Take any cube and place it in four corners, no two of which are in the same row, line, column, or great diagonal (e. g. B, C, E, H in Fig. 38), and in the four remaining corners place its complementary cube. The skeleton is now complete, and the cube may be formed from the odd magic of half its order.

This method we shall not follow further, but shall now turn to the consideration of index-cubes, an artifice far preferable.

Before proceeding the reader should carefully study the method of the index-rod as used for magic squares (pp. 57-61).

The reversion of a pair of planes in each of the three

aspects, as previously employed for 4^3 , is evidently equivalent to interchanging two numbers with their complementaries in every row, line, and column of the natural cube. If therefore we define the symbol \times as implying that such an interchange is to be made not only from the cell in which it is placed, but also from the three other cells with which it is symmetrically situated in regard to the central horizontal and vertical bars of its P-plane, and can make

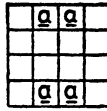


Fig. 24.

one such symbol operate in every row, line and column of an index-cube whose edge is half that of the great cube, we shall have secured the equivalent of the above-mentioned reversion. For example, a \times placed in the second cell of the top row of any P-plane of 4^3 , will denote that the four numbers marked *a* in Fig. 24 are each to be interchanged with its complement, which lies in the associated cell in the associated P-plane.

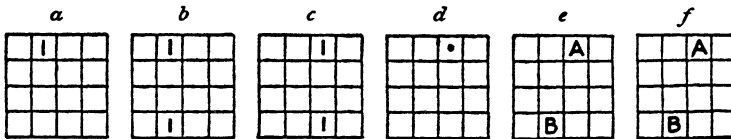


FIG. 25.

From this it follows that we shall have a complete reversion scheme for any order $4m$, by placing in every row, line and column of the index $(2m)^3$, m of the symbols \times . In the case of orders $4m+2$, after placing m such symbols in the cube $(2m+1)^3$, we have still to make the equivalent of one reversed plane in each of the three aspects. This amounts to making one symmetrical vertical interchange, one symmetrical horizontal interchange, and one

symmetrical interchange at right angles to the paper-plane in every row, line and column. If we use the symbol $|$ to denote such a vertical interchange, not only for the cell in which it stands, but also for the associated cell, and give like meanings to $—$ and \cdot , for horizontal changes and changes along lines, we shall have made the broken reversion when we allow each of these symbols to operate once in every row, column and line of the index. For example, a in Fig. 25 means b in its own P-plane, and c in the associated P-plane; while d indicates that the numbers lying in its own P-plane as in e are to be interchanged, A with A and B with B, with the numbers lying in the associated plane f . We can always prepare the index, provided the rod does not contain a less number of cells than the number of symbols, by the following rule, n being the order.

Take an index-rod of $n/2$ cells, $n/4$ symbols of the form \times , (using the integral part of $n/4$ only), and if there is any remainder when n is divided by 4 add the three symbols $|$, $—$, \cdot . Now prepare an index square in the way described on p. 59, but using the diagonals upward and to the right instead of upward to the left,⁷ and take this square as the first P-plane of an index-cube. Fill every *great* diagonal of the cube, running to the *right, down* and *away*, which has a symbol in this P-plane cell, with repetitions of that symbol.⁸ This index-cube applied to the near, left-hand, top corner of the natural n^3 , with the symbols allowed the operative powers already defined, will make the magic n^3 .

This method for even orders applies universally with the single exception of 6^3 , and in the case of 6^3 we shall presently show that the broken reversion can still be made

⁷ Either way will do, but it happens that the former has been used in the examples which follow.

⁸ More briefly, in the language of Paths, the symbols are laid, in the square, on $(1, 1)$; their repetitions in the cube, on $(1, —1, 1)$.

by scattering the symbols over the whole cube. The following are index-rods for various cubes.

4^3	<table><tr><td>×</td><td></td></tr></table>	×		12^3	<table><tr><td></td><td></td><td>×</td><td>×</td><td></td><td>×</td></tr></table>			×	×		×			
×														
		×	×		×									
8^3	<table><tr><td>×</td><td></td><td></td><td>×</td></tr></table>	×			×	14^3	<table><tr><td></td><td>×</td><td>-</td><td>×</td><td>•</td><td>×</td><td>1</td></tr></table>		×	-	×	•	×	1
×			×											
	×	-	×	•	×	1								
10^3	<table><tr><td>×</td><td>1</td><td>-</td><td>×</td><td>•</td></tr></table>	×	1	-	×	•								
×	1	-	×	•										

As in the case of index-rods for squares, the first cell may be left blank, otherwise it must contain a ×.

I	II	III	IV																																																																
<table><tr><td>64</td><td>2</td><td>3</td><td>61</td></tr><tr><td>5</td><td>59</td><td>58</td><td>8</td></tr><tr><td>9</td><td>55</td><td>54</td><td>12</td></tr><tr><td>52</td><td>14</td><td>15</td><td>49</td></tr></table>	64	2	3	61	5	59	58	8	9	55	54	12	52	14	15	49	<table><tr><td>48</td><td>18</td><td>19</td><td>45</td></tr><tr><td>21</td><td>43</td><td>42</td><td>24</td></tr><tr><td>25</td><td>39</td><td>38</td><td>28</td></tr><tr><td>36</td><td>30</td><td>31</td><td>33</td></tr></table>	48	18	19	45	21	43	42	24	25	39	38	28	36	30	31	33	<table><tr><td>32</td><td>34</td><td>35</td><td>29</td></tr><tr><td>37</td><td>27</td><td>26</td><td>40</td></tr><tr><td>41</td><td>23</td><td>22</td><td>44</td></tr><tr><td>20</td><td>46</td><td>47</td><td>17</td></tr></table>	32	34	35	29	37	27	26	40	41	23	22	44	20	46	47	17	<table><tr><td>16</td><td>50</td><td>51</td><td>13</td></tr><tr><td>53</td><td>11</td><td>10</td><td>56</td></tr><tr><td>57</td><td>7</td><td>6</td><td>60</td></tr><tr><td>4</td><td>62</td><td>63</td><td>1</td></tr></table>	16	50	51	13	53	11	10	56	57	7	6	60	4	62	63	1
64	2	3	61																																																																
5	59	58	8																																																																
9	55	54	12																																																																
52	14	15	49																																																																
48	18	19	45																																																																
21	43	42	24																																																																
25	39	38	28																																																																
36	30	31	33																																																																
32	34	35	29																																																																
37	27	26	40																																																																
41	23	22	44																																																																
20	46	47	17																																																																
16	50	51	13																																																																
53	11	10	56																																																																
57	7	6	60																																																																
4	62	63	1																																																																

Fig. 26.

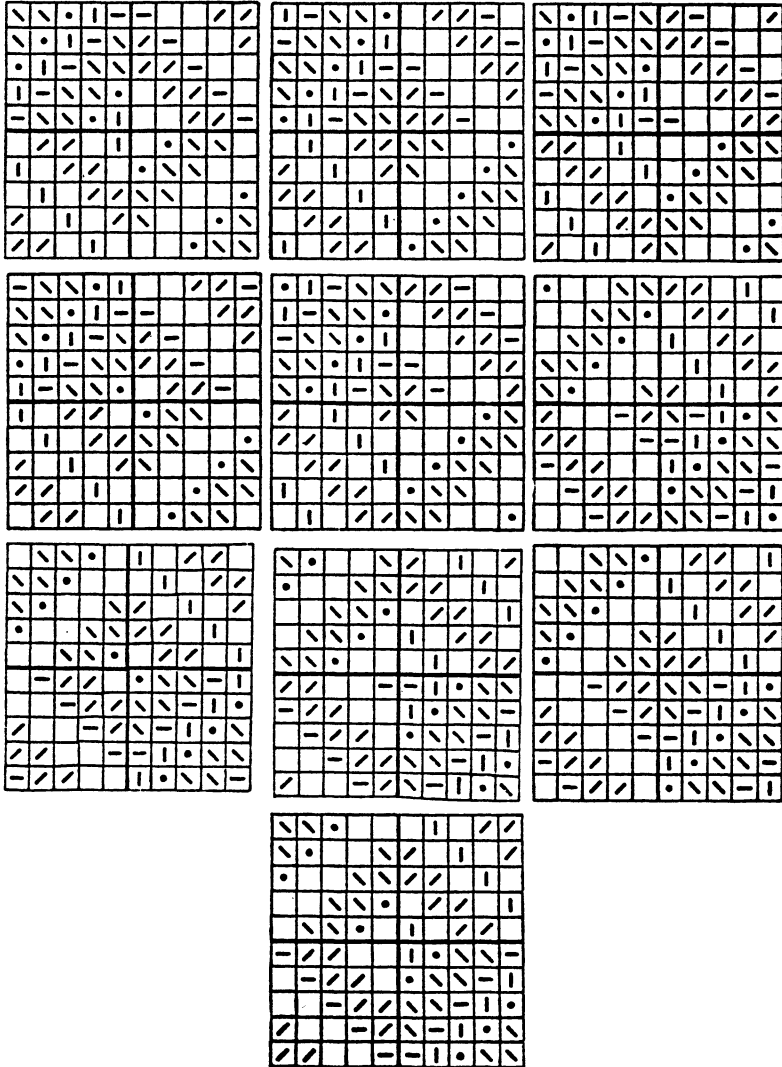
Fig. 26 is a 4^3 , made with the index-rod given above. It has only half the numbers removed from their natural places. Figs. 27 and 28 are the index-rod, index-square and index-cube for 10^3 , and Fig. 29 is the extended reversion scheme obtained from these, in which \ and / denote single changes between associated cells, and the symbols |, —, and •, single changes parallel to columns, rows, and lines. Figs. 30 and 31 show the resulting cube.

<table><tr><td>x</td><td>x</td><td>•</td><td>1</td><td>-</td></tr></table>	x	x	•	1	-	<table><tr><td>x</td><td>x</td><td>•</td><td>1</td><td>-</td></tr><tr><td>x</td><td>•</td><td>1</td><td>-</td><td>x</td></tr><tr><td>•</td><td>1</td><td>-</td><td>x</td><td>x</td></tr><tr><td>1</td><td>-</td><td>x</td><td>x</td><td>•</td></tr><tr><td>-</td><td>x</td><td>x</td><td>•</td><td>1</td></tr></table>	x	x	•	1	-	x	•	1	-	x	•	1	-	x	x	1	-	x	x	•	-	x	x	•	1
x	x	•	1	-																											
x	x	•	1	-																											
x	•	1	-	x																											
•	1	-	x	x																											
1	-	x	x	•																											
-	x	x	•	1																											
Index Rod.	Index Square.																														

Fig. 27.

<table><tr><td>×</td><td>×</td><td>•</td><td>1</td><td>-</td></tr><tr><td>×</td><td>•</td><td>1</td><td>-</td><td>×</td></tr><tr><td>•</td><td>1</td><td>-</td><td>×</td><td>×</td></tr><tr><td>1</td><td>-</td><td>×</td><td>×</td><td>•</td></tr><tr><td>-</td><td>×</td><td>×</td><td>•</td><td>1</td></tr></table>	×	×	•	1	-	×	•	1	-	×	•	1	-	×	×	1	-	×	×	•	-	×	×	•	1	<table><tr><td>1</td><td>-</td><td>×</td><td>×</td><td>•</td></tr><tr><td>-</td><td>×</td><td>×</td><td>•</td><td>1</td></tr><tr><td>×</td><td>×</td><td>•</td><td>1</td><td>-</td></tr><tr><td>×</td><td>•</td><td>1</td><td>-</td><td>×</td></tr><tr><td>•</td><td>1</td><td>-</td><td>×</td><td>×</td></tr></table>	1	-	×	×	•	-	×	×	•	1	×	×	•	1	-	×	•	1	-	×	•	1	-	×	×	<table><tr><td>×</td><td>•</td><td>1</td><td>-</td><td>×</td></tr><tr><td>•</td><td>1</td><td>-</td><td>×</td><td>×</td></tr><tr><td>1</td><td>-</td><td>×</td><td>×</td><td>•</td></tr><tr><td>-</td><td>×</td><td>×</td><td>•</td><td>1</td></tr><tr><td>×</td><td>×</td><td>•</td><td>1</td><td>-</td></tr></table>	×	•	1	-	×	•	1	-	×	×	1	-	×	×	•	-	×	×	•	1	×	×	•	1	-	<table><tr><td>-</td><td>×</td><td>×</td><td>•</td><td>1</td></tr><tr><td>×</td><td>×</td><td>•</td><td>1</td><td>-</td></tr><tr><td>×</td><td>•</td><td>1</td><td>-</td><td>×</td></tr><tr><td>•</td><td>1</td><td>-</td><td>×</td><td>×</td></tr><tr><td>1</td><td>-</td><td>×</td><td>×</td><td>•</td></tr></table>	-	×	×	•	1	×	×	•	1	-	×	•	1	-	×	•	1	-	×	×	1	-	×	×	•	<table><tr><td>•</td><td>1</td><td>-</td><td>×</td><td>×</td></tr><tr><td>1</td><td>-</td><td>×</td><td>×</td><td>•</td></tr><tr><td>-</td><td>×</td><td>×</td><td>•</td><td>1</td></tr><tr><td>×</td><td>×</td><td>•</td><td>1</td><td>-</td></tr><tr><td>•</td><td>1</td><td>-</td><td>×</td><td>×</td></tr></table>	•	1	-	×	×	1	-	×	×	•	-	×	×	•	1	×	×	•	1	-	•	1	-	×	×
×	×	•	1	-																																																																																																																													
×	•	1	-	×																																																																																																																													
•	1	-	×	×																																																																																																																													
1	-	×	×	•																																																																																																																													
-	×	×	•	1																																																																																																																													
1	-	×	×	•																																																																																																																													
-	×	×	•	1																																																																																																																													
×	×	•	1	-																																																																																																																													
×	•	1	-	×																																																																																																																													
•	1	-	×	×																																																																																																																													
×	•	1	-	×																																																																																																																													
•	1	-	×	×																																																																																																																													
1	-	×	×	•																																																																																																																													
-	×	×	•	1																																																																																																																													
×	×	•	1	-																																																																																																																													
-	×	×	•	1																																																																																																																													
×	×	•	1	-																																																																																																																													
×	•	1	-	×																																																																																																																													
•	1	-	×	×																																																																																																																													
1	-	×	×	•																																																																																																																													
•	1	-	×	×																																																																																																																													
1	-	×	×	•																																																																																																																													
-	×	×	•	1																																																																																																																													
×	×	•	1	-																																																																																																																													
•	1	-	×	×																																																																																																																													

Fig. 28. Index Cube.

Fig. 29. Extended Reversion Scheme for 10^6 .

100	999	903	94	6	5	7	8	992	991
990	912	83	17	986	985	14	18	19	981
921	72	28	977	976	975	974	23	29	30
61	39	968	967	935	36	964	963	32	40
50	959	958	944	55	46	47	953	952	41
51	949	948	54	45	56	957	943	942	60
31	62	938	937	65	966	934	933	69	70
71	22	73	927	926	925	924	78	79	980
920	82	13	84	916	915	87	88	989	911
910	909	93	4	95	96	97	998	902	901

191	109	898	897	805	106	894	893	102	110
120	889	888	814	185	116	117	883	882	111
880	879	823	174	126	125	127	128	872	871
870	832	163	137	866	865	134	138	139	861
841	152	148	857	856	855	854	143	149	150
151	142	153	847	846	845	844	158	159	860
840	162	133	164	836	835	167	168	869	831
830	829	173	124	175	176	177	878	822	821
181	819	818	184	115	186	887	813	812	190
101	192	808	807	195	896	804	803	199	200

800	702	293	207	796	795	204	208	209	791
711	282	218	787	786	785	784	213	219	220
271	229	778	777	725	226	774	773	222	230
240	769	768	734	265	236	237	763	762	231
760	759	743	254	246	245	247	248	752	751
750	749	253	244	255	256	257	758	742	741
261	739	738	264	235	266	767	733	732	270
221	272	728	727	275	776	724	723	279	280
281	212	283	717	716	715	714	288	289	790
710	292	203	294	706	705	297	298	799	701

310	699	698	604	395	306	307	693	692	301
690	689	613	384	316	315	317	318	682	681
680	622	373	327	676	675	324	328	329	671
631	362	338	667	666	665	664	333	339	340
351	349	658	657	645	346	654	653	342	350
341	352	648	647	355	656	644	643	359	360
361	332	363	637	636	635	634	368	369	670
630	372	323	374	626	625	377	378	679	621
620	619	383	314	385	386	387	688	612	611
391	609	608	394	305	396	697	603	602	400

501	492	408	597	596	595	594	403	409	410
481	419	588	587	515	416	584	583	412	420
430	579	578	524	475	426	427	573	572	421
570	569	533	464	436	435	437	438	562	561
560	542	453	447	556	555	444	448	449	551
550	452	443	454	546	545	457	458	559	541
540	539	463	434	465	466	467	568	532	531
471	529	528	474	425	476	577	523	522	480
411	482	518	517	485	586	514	513	489	490
491	402	493	507	506	505	504	498	499	600

401	502	503	497	496	495	494	508	599	510
511	512	488	487	415	516	484	483	519	590
521	479	478	424	525	576	527	473	472	530
470	469	433	534	535	536	567	538	462	461
460	442	543	544	456	455	547	558	549	451
450	552	553	557	446	445	554	548	459	441
440	439	563	564	566	565	537	468	432	431
580	429	428	574	575	526	477	423	422	571
581	589	418	417	585	486	414	413	582	520
591	592	598	407	406	405	404	593	509	500

Fig. 30. First 6 plates of 10^6 , made from Fig. 29. (Sum = 5005.)

601	399	398	304	605	696	607	393	392	610
390	389	313	614	615	616	687	618	382	381
380	322	623	624	376	375	627	678	629	371
331	632	633	367	366	365	364	638	669	640
641	642	358	357	345	646	354	353	649	660
651	659	348	347	655	356	344	343	652	650
661	662	668	337	336	335	334	663	639	370
330	672	673	677	326	325	674	628	379	321
320	319	683	684	686	685	617	388	312	311
700	309	308	694	695	606	395	303	302	691

300	202	703	704	296	295	707	798	709	291
211	712	713	287	286	285	284	718	789	720
721	722	278	277	225	726	274	273	729	780
731	269	268	234	735	766	737	263	262	740
260	259	243	744	745	746	757	748	252	251
250	249	753	754	756	755	747	258	242	241
770	239	238	764	765	736	267	233	232	761
771	779	228	227	775	276	224	223	772	730
781	782	788	217	216	215	214	783	719	290
210	792	793	797	206	205	794	708	299	201

801	802	198	197	105	806	194	193	809	900
811	189	188	114	815	886	817	183	182	820
180	179	123	824	825	826	877	828	172	171
170	132	833	834	166	165	837	868	839	161
141	842	843	157	156	155	154	848	859	850
851	852	858	147	146	145	144	853	849	160
140	862	863	867	136	135	864	838	169	131
130	129	873	874	876	875	827	178	122	121
890	119	118	884	885	816	187	113	112	881
891	899	108	107	895	196	104	103	892	810

100	99	3	904	905	906	997	908	92	91
90	12	913	914	86	85	917	988	919	81
21	922	923	77	76	75	74	928	979	930
931	932	68	67	35	936	64	63	939	970
941	59	58	44	945	956	947	53	52	950
960	49	48	954	955	946	57	43	42	951
961	969	38	37	965	66	34	33	962	940
971	972	978	27	26	25	24	973	929	80
20	982	983	987	16	15	984	918	89	11
10	9	993	994	996	995	907	98	2	1

Fig. 31. Last 4 plates of 10^8 , made from Fig. 29. (Sum = 5005.)

If we attack 63 by the general rule, we find 4 symbols, \times , —, |, ·, and only 3 cells in the rod; the construction is therefore impossible. Suppose we construct an index-cube from the rod $\boxed{\times} \boxed{1} \boxed{-}$, we shall find it impossible to distribute the remaining symbol $\boxed{\cdot}$ in the extended reversion-scheme obtained from this index. The feat, however, is possible if we make (for this case only) a slight change in the meanings of | and —. By the general rule \times operates on 4 cells in its own P-plane, where, by the rule of association,

the planes are paired thus: $\left| \begin{array}{ccc} 1 & \text{with} & 6 \\ 2 & \text{"} & 5 \\ 3 & \text{"} & 4 \end{array} \right|$. In interpreting

the meanings of | and —, in this special case, we must make a cyclic change in the right-hand column of this little table.

Thus for “|” $\begin{vmatrix} 1 & \text{with } 5 \\ 2 & \text{“ } 4 \\ 3 & \text{“ } 6 \end{vmatrix}$, and for “—” $\begin{vmatrix} 1 & \text{with } 4 \\ 2 & \text{“ } 6 \\ 3 & \text{“ } 5 \end{vmatrix}$. This

means that a [1], for example, in the second P-plane has its usual meaning in that plane, and also acts on the two cells which would be the associated cells if the 4th plane were to become the 5th, etc. If we extend this scheme, there will be just room to properly distribute the [1]’s in the two parallelipeds which form the right-hand upper and left-hand lower quarters of the cube, as shown in Fig. 32.

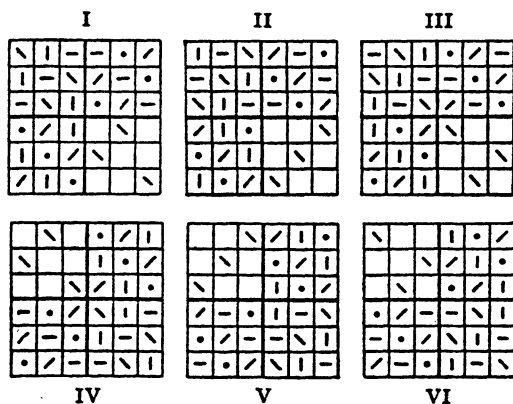


Fig. 32. Extended Reversion-Scheme for 6^3 .

This scheme produces the cube shown below, which is magic on its 36 rows, 36 columns, 36 lines, *and on its 4 central great diagonals.*

Fig. 32 is the identical scheme discovered by Firth in 1889, and was obtained a few months later than the pseudo-skeleton shown in Fig. 21. A year or two earlier he had discovered the broken reversion for squares of even order, but he never generalized the method, or conceived the idea of an index-cube. The development of the method as here described was worked out by the present writer in 1894.

About the same time Rouse Ball, of Trinity College, Cambridge, independently arrived at the method of reversions for squares (compare the earlier editions of his *Mathematical Recreations*, Macmillan), and in the last edition, 1905, he adopts the idea of an index-square; but he makes no application to cubes or higher dimensions. There is reason to believe, however, that the idea of reversions by means of an index-square was known to Fermat. In his letter to

I						II						III					
216	32	4	3	185	211	67	41	178	177	38	150	78	143	105	112	140	73
25	11	208	207	8	192	48	173	63	154	170	43	138	98	82	81	119	133
18	203	21	196	200	13	168	56	52	51	161	163	91	89	130	129	86	126
199	197	15	22	194	24	162	50	165	58	59	157	85	128	124	123	95	96
7	206	190	189	29	30	169	155	45	64	152	66	120	80	135	100	101	115
186	2	213	34	35	181	37	176	148	147	71	72	139	113	75	106	110	108

IV						V						VI					
109	107	111	76	104	144	145	146	70	69	179	42	36	182	183	214	5	31
102	116	117	136	83	97	151	65	153	46	62	174	187	188	28	27	209	12
121	122	94	93	131	90	60	158	159	166	53	55	193	23	195	16	20	204
132	92	88	87	125	127	54	167	57	160	164	49	19	17	202	201	14	198
84	137	99	118	134	79	61	47	172	171	44	156	210	26	10	9	191	205
103	77	142	141	74	114	180	68	40	39	149	175	6	215	33	184	212	1

Fig. 33, made from Fig. 32. Sum = 651.

Mersenne of April 1, 1640, (*Œuvres de Fermat*, Vol. II, p. 193), he gives the square of order 6 shown in Fig. 34. This is obtained by applying the index (Fig. 35) to the *bottom* left-hand corner of the natural square written from below upwards, i. e., with the numbers 1 to 6 in the bottom row, 7 to 12 in the row above this, etc. There is nothing surprising in this method of writing the natural square, in fact it is suggested by the conventions of Cartesian geometry, with which Fermat was familiar. There is a

much later similar instance: Cayley, in 1890, dealing with "Latin squares," writes from below upwards, although Euler, in his original *Memoire* (1782), wrote from above downwards. Another square of order 6, given by Fermat, in the same place, is made from the same index, but is disguised because he uses a "deformed" natural square.

6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
25	29	10	9	26	12
36	5	33	4	2	31

Fig. 34.

—	1	x
1	x	—
x	—	1

Fig. 35.

It is interesting to note that all these reversion magics (unlike those made by Thompson's method), are La Hireian, and also that the La Hireian scheme can be obtained by turning a single outline on itself. To explain this statement we will translate the square in Fig. 12 into the scale

A	55	04	52	03	01	50
	40	44	13	12	41	15
	25	31	33	32	24	20
	30	21	23	22	34	35
	10	14	42	43	11	45
	05	51	02	53	54	00
						B

Fig. 36.

whose radix is 6, first decreasing every number by unity. This last artifice is merely equivalent to using the n^2 consecutive numbers from 0 to n^2-1 , instead of from 1 to n^2 , and is convenient because it brings the scheme of units and the scheme of 6's digits into uniformity.

If we examine this result as shown in Fig. 36 we

find that the scheme for units can be converted into that for the 6's, by turning the skeleton through 180° about the axis AB; that is to say, a single outline turned upon itself will produce the magic.

I						II						III					
555	051	003	002	504	550	150	104	453	452	101	405	205	354	252	303	351	200
040	014	543	542	011	515	115	444	142	413	441	110	345	241	213	212	314	340
025	534	032	523	531	020	435	131	123	122	424	430	230	224	333	332	221	325
530	524	022	033	521	035	425	121	432	133	134	420	220	331	323	322	234	235
010	541	513	512	044	045	440	414	112	143	411	145	315	211	342	243	244	310
505	001	552	053	054	500	100	451	403	402	154	155	350	304	202	253	301	255

300	254	302	203	251	355	400	401	153	152	454	105	055	501	502	553	004	050
245	311	312	343	214	240	410	144	412	113	141	445	510	511	043	042	544	015
320	321	233	232	334	225	135	421	422	433	124	130	520	034	522	023	031	535
335	231	223	222	324	330	125	434	132	423	431	120	030	024	533	532	021	525
215	344	242	313	341	210	140	114	443	442	111	415	545	041	013	012	514	540
250	204	353	352	201	305	455	151	103	102	404	450	005	554	052	503	551	000

Fig. 37.

The same is true of the cube; that is, just as we can obtain a La Hirean scheme for a square by turning a single square outline once upon itself, so a similar scheme for a cube can be obtained by turning a cubic outline

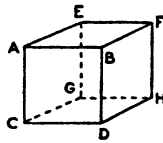


Fig. 38.

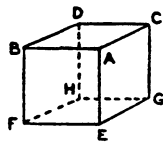


Fig. 39.

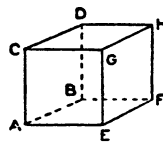


Fig. 40.

twice upon itself. If we reduce all the numbers in Fig. 33 by unity and then "unroll" the cube, we get the La Hirean scheme of Fig. 37 in the scale radix 6.

If now we represent the skeleton of the 6's: (left-hand) digits by Fig. 38, and give this cube the "twist" indicated

by Fig. 39, we shall get the skeleton of the 6's (middle) digits, and the turn suggested by Fig. 40 gives that of the units (right-hand) digits. Thus a single outline turned twice upon itself gives the scheme.

We can construct any crude magic octahedroid⁹ of

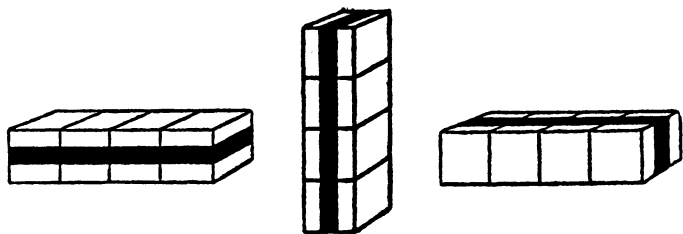


Fig. 41, 1st reversion. Fig. 42, 2d reversion. Fig. 43, 3d reversion.

double-of-even order, by the method of reversions, as shown with 4⁴ in Figs. 41 to 44.

The first three reversions will be easily understood from the figures, but the fourth requires some explanation. It actually amounts to an interchange between every pair



Fig. 44, 4th reversion.

of numbers in associated cells of the parallelopiped formed by the two central cubical sections. If the reader will use a box or some other "rectangular" solid as a model, and number the 8 corners, he will find that such a change cannot be effected in three-dimensional space by turning the

DIMENSIONS	REGULAR FIGURE	BOUNDARIES
2	<i>Tetragon</i> (or square)	4 one-dimensional straight lines
3	<i>Hexahedron</i> (cube)	6 two-dimensional squares
4	<i>Octahedroid</i>	8 three-dimensional cubes
etc.	etc.	etc.

parallelopiped as a whole, on the same principle that a right hand cannot, by any turn, be converted into a left hand. But such a change can be produced by a single turn in 4-dimensional space; in fact this last reversion is made with regard to an axis in the 4th, or imaginary direction.

1	2	3	4	65	66	67	68	129	130	131	132	193	194	195	196
248	247	246	245	184	183	182	181	120	119	118	117	56	55	54	53
252	251	250	249	188	187	186	185	124	123	122	121	60	59	58	57
13	14	15	16	77	78	79	80	141	142	143	144	205	206	207	208

17	18	19	20	81	82	83	84	145	146	147	148	209	210	211	212
232	231	230	229	168	167	166	165	104	103	102	101	40	39	38	37
236	235	234	233	172	171	170	169	108	107	106	105	44	43	42	41
29	30	31	32	93	94	95	96	157	158	159	160	221	222	223	224

33	34	35	36	97	98	99	100	161	162	163	164	225	226	227	228
216	215	214	213	152	151	150	149	88	87	86	85	24	23	22	21
220	219	218	217	156	155	154	153	92	91	90	89	28	27	26	25
45	46	47	48	109	110	111	112	173	174	175	176	237	238	239	240

49	50	51	52	113	114	115	116	177	178	179	180	241	242	243	244
200	199	198	197	136	135	134	133	72	71	70	69	8	7	6	5
204	203	202	201	140	139	138	137	76	75	74	73	12	11	10	9
61	62	63	64	125	126	127	128	189	190	191	192	253	254	255	256

Fig. 45.

The following four figures (45-48) show each stage of the process, and if the reader will compare them with the results of a like series of reversions made from a different aspect of the natural octahedroid, he will find that the "imaginary" reversion then becomes a real reversion, while

one of the reversions which was real becomes imaginary. Fig. 45 is the natural 4^4 after the first reversion, magic in columns only; Fig. 46 is Fig. 45 after the second reversion, magic in rows and columns; Fig. 47 is Fig. 46 after the third reversion, magic in rows, columns and lines; and

1	254	255	4
248	11	10	245
252	7	6	249
13	242	243	16

65	190	191	68
184	75	74	181
188	71	70	185
77	178	179	80

129	126	127	132
120	139	138	117
124	135	134	121
141	114	115	144

193	62	63	196
56	203	202	53
60	199	198	57
205	50	51	208

17	238	239	20
232	27	26	229
236	23	22	233
29	226	227	32
81	174	175	84
168	91	90	165
172	87	86	169
93	162	163	96
145	110	111	148
104	155	154	101
108	151	150	105
157	98	99	160
209	46	47	212
40	219	218	37
44	215	214	41
221	34	35	224
33	222	223	36
216	43	42	213
220	39	38	217
45	210	211	48
97	158	159	100
152	107	106	149
156	103	102	153
109	146	147	112
161	94	95	164
88	171	170	85
92	167	166	89
173	82	83	176
225	30	31	228
24	235	234	21
28	231	230	25
237	18	19	240
49	206	207	52
200	59	58	197
204	55	54	201
61	194	195	64
113	142	143	116
136	123	122	133
140	119	118	137
125	130	131	128
177	78	79	180
72	187	186	69
76	183	182	73
189	66	67	192
241	14	15	244
8	251	250	5
12	247	246	9
253	2	3	256

Fig. 46.

Fig. 48 is Fig. 47 after the fourth reversion, magic in rows, columns, lines and i 's, = crude magic 4^4 . The symbol i denotes series of cells parallel to the imaginary edge.

Fig. 48 is magic on its 64 rows, 64 columns, 64 lines, and 64 i 's, and on its 8 central hyperdiagonals. Through-

out the above operations the columns of squares have been taken as forming the four cells of the P_1 -aspect;¹⁰ the rows of squares taken to form cubes, of course, show the P_2 -aspect.

1 254 255 4	65 190 191 68	129 126 127 132	193 62 63 196
248 11 10 245	184 75 74 181	120 139 138 117	56 203 202 53
252 7 6 249	188 71 70 185	124 135 134 121	60 199 198 57
13 242 243 16	77 178 179 80	141 114 115 144	205 50 51 208

224 35 34 221	160 99 98 157	96 163 162 93	32 227 226 29
41 214 215 44	105 150 151 108	169 86 87 172	233 22 23 236
37 218 219 40	101 154 155 104	165 90 91 168	229 26 27 232
212 47 46 209	148 111 110 145	84 175 174 81	20 239 238 17

240 19 18 237	176 83 82 173	112 147 146 109	48 211 210 45
25 230 231 28	89 166 167 92	153 102 103 156	217 38 39 220
21 234 235 24	85 170 171 88	149 106 107 152	213 42 43 216
228 31 30 225	164 95 94 161	100 159 158 97	36 223 222 33

49 206 207 52	113 142 143 116	177 78 79 180	241 14 15 244
200 59 58 197	136 123 122 133	72 187 186 69	8 251 250 5
204 55 54 201	140 119 118 137	76 183 182 73	12 247 246 9
61 194 195 64	125 130 131 128	189 66 67 192	253 2 3 256

Fig. 47.

This construction has been introduced merely to accentuate the analogy between magics of various dimensions; we might have obtained the magic 4^4 much more

¹⁰ Since the 4th dimension is the square of the second, two aspects of the octahedroid are shown in the presentation plane. The 3d and 4th aspects are in H-planes and V-planes. Since there are two P-plane aspects it might appear that each would produce a different H-plane and V-plane aspect; but this is a delusion.

rapidly by a method analogous to that used for 4^3 (Fig. 26). We have simply to interchange each number in the natural octahedroid occupying a cell marked \times in Fig. 49, with its complementary number lying in the associated cell

1 254 255 4	192 67 66 189	128 131 130 125	193 62 63 196
248 11 10 245	73 182 183 76	137 118 119 140	56 203 202 53
252 7 6 249	69 186 187 72	133 122 123 136	60 199 198 57
13 242 243 16	180 79 78 177	116 143 142 113	205 50 51 208

224 35 34 221	97 158 159 100	161 94 95 164	32 227 226 29
41 214 215 44	152 107 106 149	88 171 170 85	233 22 23 236
37 218 219 40	156 103 102 153	92 167 166 89	229 26 27 232
212 47 46 209	109 146 147 112	173 82 83 176	20 239 238 17

240 19 18 237	81 174 175 84	145 110 111 148	48 211 210 45
25 230 231 28	168 91 90 165	104 155 154 101	217 38 39 220
21 234 235 24	172 87 86 169	108 151 150 105	213 42 43 216
228 31 30 225	93 162 163 96	157 98 99 160	36 223 222 33

49 206 207 52	144 115 114 141	80 179 178 77	241 14 15 244
200 59 58 197	121 134 135 124	185 70 71 188	8 251 250 5
204 55 54 201	117 138 139 120	181 74 75 184	12 247 246 9
61 194 195 64	132 127 126 129	68 191 190 65	253 2 3 256

Fig. 48.

of the associated cube. Fig. 49 is the extended skew-reversion scheme from the index-rod $\square \times$.

All magic octahedroids of double-of-odd order > 104 can be constructed by the index-rod, for just as we construct an index-square from the rod, and an index-cube from the square, so we can construct an index-octahedroid

from the cube. The magics 6^4 and 10^4 have not the capacity for construction by the general rule, but they may be

	x	x		x			x	x			x		x	x	
x			x		x	x			x	x		x			x
x			x		x	x			x	x		x			x
	x	x		x			x	x			x		x	x	
x			x		x	x			x	x		x		x	x
	x	x		x			x	x			x		x	x	
	x	x		x			x	x			x		x	x	
x			x		x	x			x	x		x			x
x			x		x	x			x	x		x			x
	x	x		x			x	x			x		x	x	
	x	x		x			x	x			x		x	x	
x			x		x	x			x	x		x			x
	x	x		x			x	x			x		x	x	
x			x		x	x			x	x		x			x
x			x		x	x			x	x		x			x
	x	x		x			x	x			x		x	x	

Fig. 49. Skew Reversion for 4^4 .

obtained by scattering the symbols over the whole figure as we did with 6^3 .

C. PLANCK.

HAYWARD'S HEATH, ENGLAND.